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Application of Shoshany-Snodgrass Analysis to the Natario Warp Drive Space time with Zero Expansion

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Supplementary

Appendix A: Differential forms, Hodge star and the mathematical demonstration of the Natario vectors nX=−vsdx and nX=vsdx for a constant speed vs in a R^3 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows (see pg 4 in [2],eqs 3.135 and 3.137 pg 82(a)(b) in [15],eq 3.72 pg 69(a)(b) in [15]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{68}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta (d\varphi \wedge dr) \tag{69}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{70}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{71}$$

$$rd\theta \sim r\sin\theta (d\varphi \wedge dr) \tag{72}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{73}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by (see pg 8 in [4],eq 3.72 pg 69(a)(b) in [15]):

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$$*dr = r^2 \sin\theta(d\theta \wedge d\varphi) \tag{74}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{75}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{76}$$

Back again to the Natario equivalence between spherical and cartezian coordinates (pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{77}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{78}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{79}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{80}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta \wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi \wedge dr)] \tag{81}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta \wedge d\varphi)] - [r\sin^2\theta(d\varphi \wedge dr)] \tag{82}$$

We know that the following expression holds true (see pg 9 in [3], eq 3.79 pg 70(a)(b) in [15]):

$$d\varphi \wedge dr = -dr \wedge d\varphi \tag{83}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta \wedge d\varphi)] + [r\sin^2\theta(dr \wedge d\varphi)] \tag{84}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates (pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{85}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{86}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{87}$$

According to pg 10 in [3],eq 3.90 pg 74(a)(b) in [15] the term 1 2 r 2 sin2 θ × d[(dϕ)]=0

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{88}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{89}$$

Because and according to pg 10 in [3], eqs 3.90 and 3.91 pg 74(a)(b) in [15],tb 3.2 pg 68(a)(b) in [15]:

$$*d(\alpha+\beta) = d\alpha + d\beta \tag{90}$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p=2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{91}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{92}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{93}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{94}$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{95}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced 1/2 by the function f(r):

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{96}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{97}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{98}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{99}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{100}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2])

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{101}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \sim -r\sin\theta(dr \wedge d\varphi) \tag{102}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{103}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta \wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr \wedge d\varphi)] + f'(r) r\sin\theta[r\sin\theta(dr \wedge d\varphi)] \tag{104}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{105}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{106}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written:

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{107}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{108}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{109}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{110}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{111}$$

$$nX = -2vs(t)f(r)\ cos\theta dr + vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{112}$$

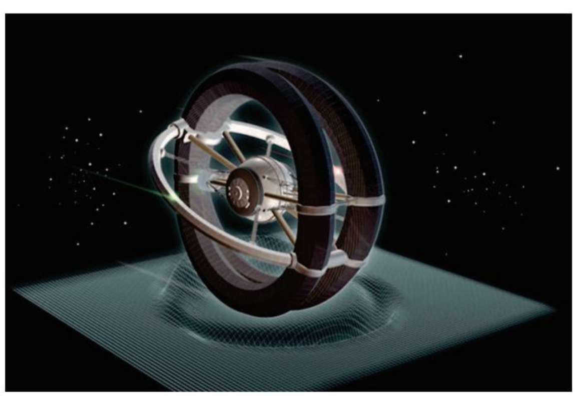

FIG. 1. Artistic representation of the Natario warp drive. Note in the bottom of the figure the Alcubierre expansion of the normal volume elements. (Source: Internet)

Appendix B: Artistic Presentation of the Natario warp drive

According to the geometry of the Natario warp drive the spacetime contraction in one direction(radial) is balanced by the spacetime expansion in the remaining direction(perpendicular). (pg 5 in [2]). The expansion of the normal volume elements in the Natario warp drive is given by the following expressions (pg 5 in [2]).

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{113}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{114}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = v_s n'(r)\cos\theta \tag{115}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{116}$$

If we expand the radial direction the perpendicular direction contracts to keep the expansion of the normal volume elements equal to zero.

This figure is a pedagogical example of the graphical presentarion of the Natario warp drive.

The ”bars” in the figure were included to illustrate how the expansion in one direction can be counter balanced by the contraction in the other directions.These ”bars” keeps the expansion of the normal volume elements in the Natario warp drive equal to zero.

Note also that the graphical presentation of the Alcubierre warp drive expansion of the normal volume elements according to fig 1 pg 10 in [1] is also included

Note also that the energy density in the Natario warp drive 3+1 spacetime being given by the following expressions (pg 5 in [2]):

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(r))^2\cos^2\theta + \left(n'(r) + \frac{r}{2}n''(r)\right)^2\sin^2\theta\right]. \tag{117}$$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2\cos^2\theta + \left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2\sin^2\theta\right]. \tag{118}$$

Is being distributed around all the space involving the ship (above the ship sin θ=1 and cos θ=0 while in front of the ship sin θ=0 and cos θ=1). The negative energy in front of the ship” deflect” photons or other particles so these will not reach the ship inside the bubble. The illustrated” bars” are the obstacles that deflects photons or incoming particles from outside the bubble never allowing these to reach the interior of the bubble [13].

The negative energy density has repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls (see pg [116(a)] [116(b)] in [26])

Energy directly above the ship (y–axis)

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\left(\frac{dn(r)}{dr} + \frac{r}{2}\frac{d^2n(r)}{dr^2}\right)^2 \sin^2\theta\right]. \tag{119}$$

Energy directly in front of the ship(x–axis)

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(r)}{dr})^2 \cos^2\theta\right]. \tag{120}$$

Note also that even in a 1 + 1 dimensional spacetime the Natario warp drive retains the zero expansion behavior:

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{121}$$

$$K_{\theta\theta} = \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{122}$$

$$K_{\varphi\varphi} = \frac{X^r}{r} = v_s n'(r)\cos\theta \tag{123}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{124}$$

In all these equations the term r is our term rs

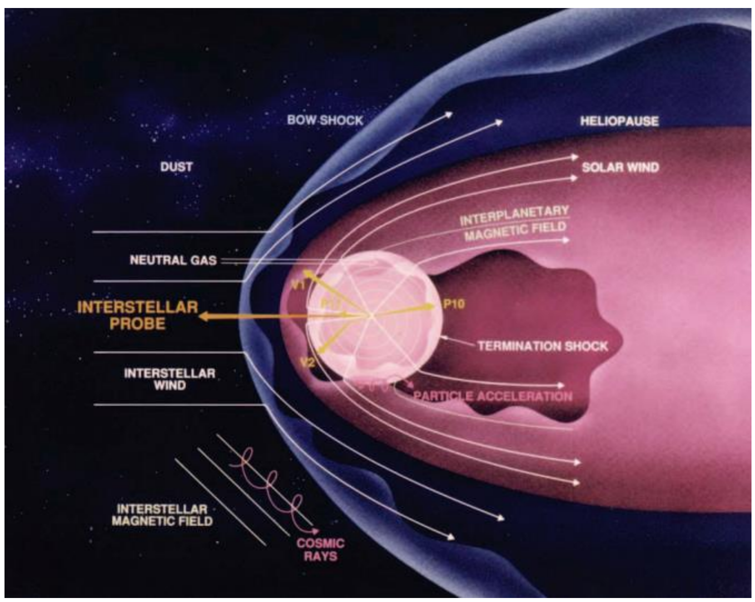

FIG. 2. Artistic representation of a Natario warp wrive in a real superluminal space travel. Note the negative energy in front of the ship deflecting incoming hazardous interstellar matter (brown arrows). (Source: Internet)

Appendix C: Artistic Presentation of a Natario warp drive in a real faster than light interstellar spaceflight

Above is being presented the artistic presentation of a Natario warp drive in a real interstellar superluminal travel. The ”ball” or the spherical shape is the Natario warp bubble with the negative energy surrounding the ship in all directions and mainly protecting the front of the bubble [14].

The brown arrows in the front of the Natario bubble are a graphical presentation of the negative energy in front of the ship deflecting interstellar dust, neutral gases, hydrogen atoms, interstellar wind photons etc. [15].

The spaceship is at the rest and in complete safety inside the Natario bubble.

In order to allow to the negative energy density of the Natario warp drive the deflection of incoming hazardous particles from the Interstellar Medium(IM) the Natario warp drive energy density must be heavier or denser when compared to the IM density. The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls (see pg [116(a)][116(b)] in [26])

Appendix D: The Natario warp drive negative energy density in Cartezian coordinates

The negative energy density according to Natario is given by (see pg 5 in [2]) [16]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \qquad (125)$$

In the bottom of pg 4 in [2] Natario defined the x-axis as the polar axis. In the top of page 5 we can see that x = rs cos(θ) implying in cos(θ)=x/rs and in sin(θ)=y/rs

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \qquad (126)$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2+z^2] = 0$ and $rs^2=[(x-xs)^2]$ and making xs=0 the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $rs^2=x^2$ because in the equatorial plane y=z=0. Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \qquad (127)$$

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls (see pg [116(a)] [116(b)] in [26])

Appendix E: Mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original 3+1 ADM Formalism according to MTW and Alcubierre

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic space time metric $ds^2=g_{\mu\nu}dx^{\mu}dx^{\nu}$ where do not exists a clear difference between space and time. This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the space time metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However, there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The 3 + 1 ADM formalism allows ourselves to separate from the generic equation $ds^2=g_{\mu\nu}dx^\mu dx^\nu$ of a given space time the 3 dimensions of space and the time dimension. (see pg [64(b)] [79(a)] in [12])

Consider a 3 dimensional hypersurface Σ_1 in an initial time t1 that evolves to a hypersurface Σ_2 in a later time t2 and hence evolves again to a hypersurface Σ_3 in an even later time t3 according to fig 2.1 pg [65(b)] [80(a)] in [12].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time t1. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σt and $\Sigma t+dt$. (see bottom of pg [65(b)] [80(a)] in [12])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg [66(b)] [81(a)] in [12]) (see also fig 21.2 pg [506(b)] [533(a)] in [11] where $dx^i+\beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}dx^\mu dx^\nu=-\alpha^2dt^2+\gamma ij\ (dx^i+\beta^i dt)\ (dx^j+\beta^j dt)$ at pg [507(b)] [534(a)] in [11]) [17].

- The 3 dimensional metric $dl^2=\gamma_{ij}dx^i dx^j$ with i, j=1, 2, 3 that measures the proper distance between two points inside each hypersurface
- The lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces (Eulerian obsxervers) $d\tau=\alpha dt$ where α is known as the lapse function.
- The relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i+\beta^i dt)$. β^i is known as the shift vector.

Combining the eqs (21.40), (21.42) and (21.44) pgs [507, 508(b)] [534, 535(a)] in [11] with the eqs (2.2.5) and (2.2.6) pgs [67(b)] [82(a)] in [12] using the signature (−, +, +, +) we get the original equations of the 3+1 ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2+\beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{128}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{129}$$

The components of the inverse metric are given by the matrix inverse:

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{130}$$

The space time metric in 3+1 is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{131}$$

But since $dl^2=\gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2=\gamma_{ii}dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{132}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{133}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{134}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{135}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i\beta^i dt^2 = \beta_i\beta^i dt^2 \tag{136}$$

$$(dx^i)^2 = dx^i dx^i \tag{137}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{138}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{139}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{140}$$

Note that the expression above is exactly the eq (2.2.4) pgs [67(b)] [82(a)] in [12]. It also appears as eq 1 pg 3 in [1].

With the original equations of the 3+1 ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{141}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{142}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i\beta^i}{\alpha^2} \end{pmatrix} \tag{143}$$

and suppressing the lapse function making α=1 we have:

$$ds^2 = (-1 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{144}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{145}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i\beta^i \end{pmatrix} \tag{146}$$

changing the signature from (−, +, +, +) to signature (+, −, −, −) we have:

$$ds^2 = -(-1 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{147}$$

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{148}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{149}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{150}$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{151}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from (−, +, +, +) to (+, −, −, −) (pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{152}$$

The Natario equation given above is valid only in cartezian coordinates. For a generic coordinates system, we must employ the equation that obeys the 3 + 1 ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{153}$$

Comparing all these equations

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{154}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{155}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{156}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{157}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{158}$$

We can see that $\beta^i = -X^i$, $\beta_i = -X_i$ and $\beta_i\beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector. Hence we have:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{159}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{160}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{161}$$

Looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{162}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{163}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{164}$$

But remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_{rs} and X_{θ} with r=rs are given by:

$$X_i = \gamma_{ii} X^i \tag{165}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = 2v_s n(rs) \cos\theta = X^r = X^{rs} \tag{166}$$

$$X_{\theta} = \gamma_{\theta\theta} X^{\theta} = rs^2 X^{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{167}$$

The equations of the Natario warp drive in the 3+1 ADM formalism are given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{168}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{169}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{170}$$

The matrix components 2 × 2 evaluated separately for rs and θ gives the following results: [18]

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0r} \\ g_{r0} & g_{rr} \end{pmatrix} = \begin{pmatrix} 1 - X_r X^r & X_r \\ X_r & -\gamma_{rr} \end{pmatrix} \quad (171)$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0r} \\ g^{r0} & g^{rr} \end{pmatrix} = \begin{pmatrix} 1 & X^r \\ X^r & -\gamma^{rr} + X^r X^r \end{pmatrix} \quad (172)$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0\theta} \\ g_{\theta 0} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 - X_\theta X^\theta & X_\theta \\ X_\theta & -\gamma_{\theta\theta} \end{pmatrix} \quad (173)$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0\theta} \\ g^{\theta 0} & g^{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & X^\theta \\ X^\theta & -\gamma^{\theta\theta} + X^\theta X^\theta \end{pmatrix} \quad (174)$$

Then the equation of the Natario warp drive spacetime with a constant speed vs in the original 3 + 1 ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \quad (175)$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drsdt + X_\theta d\theta dt) - drs^2 - rs^2 d\theta^2 \quad (176)$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \quad (177)$$

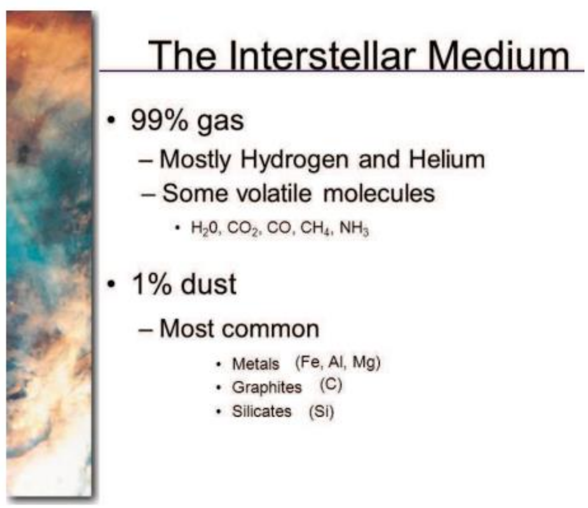

FIG. 3. Composition of the Interstellar Medium IM (Source: Internet).

Appendix F: Composition of the Interstellar Medium IM

Appendix G: Dimensional Reduction from c^4/G to c^2/G

The Alcubierre expressions for the Negative Energy Density in Geometrized Units c=G=1 are given by (pg 4 in [2]) (pg 8 in [1]):19

$$\rho = -\frac{1}{32\pi} vs^2 \left[f'(rs)\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{178}$$

$$\rho = -\frac{1}{32\pi} vs^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{179}$$

In this system all physical quantities are identified with geometrical entities such as lengths, areas or dimensionless factors. Even time is interpreted as the distance travelled by a pulse of light during that time interval, so even time is given in lengths. Energy, Momentum and Mass also have the dimensions of lengths. We can multiply a mass in kilograms by the conversion factor G/c^2 to obtain the mass equivalent in meters. On the other hand we can multiply meters by c^2/G to obtain kilograms. The energy density ($Joules/meters^3$) in Geometrized Units have a dimension of $1/length^2$ and the conversion factor for energy density is G/c^4. Again on the other hand by multiplying $1/length^2$ by c^4/G we retrieve again ($Joules/meters^3$) [20].

This is the reason why in geometrized units the Einstein tensor have the same dimension of the Stress Energy Momentum Tensor (in this case the negative energy density) and since the Einstein Tensor is associated to the Curvature of Space time both have the dimension of $1/length^2$.

$$G_{00} = 8\pi T_{00} \tag{180}$$

Passing to normal units and computing the negative energy density we multiply the Einstein Tensor (dimension $1/length^2$) by the conversion factor c^4/G in order to retrieve the normal unit for the Negative Energy Density ($Joules/meters^3$)

$$T_{00} = \frac{c^4}{8\pi G} G_{00} \tag{181}$$

Examine now the Alcubierre equations:

$vs=dxs/dt$ is dimensionless since time is also in lengths. y^2+z^2/rs^2 is dimensionless since both are given also in lengths. $f(rs)$ is dimensionless but its derivative $df(rs)/drs$ is not because rs is in meters. So the dimensional factor in Geometrized Units for the Alcubierre Energy Density comes from the square of the derivative and is also $1/length^2$. Remember that the speed of the Warp Bubble vs is dimensionless in Geometrized Units and when we multiply directly $1/length^2$ from the negative energy density in geometrized units by c^4/G to obtain the Negative Energy Density in normal units $Joules/meters^3$ the first attempt would be to make the following:

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi} vs^2 \left[f'(rs)\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{182}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi} vs^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{183}$$

But note that in normal units vs is not dimensionless and the equations above do not lead to the correct dimensionality of the Negative Energy Density because the equations above in normal units are being affected by the dimensionality of vs.

In order to make vs dimensionless again, the Negative Energy Density is written as follows:

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{184}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{185}$$

$$\rho = -\frac{c^2}{G}\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{186}$$

$$\rho = -\frac{c^2}{G}\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{187}$$

As already seen. The same results are valid for the Natario energy density

Note that from

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{188}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{189}$$

Making c=G=1 we retrieve again

$$\rho = -\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{190}$$

$$\rho = -\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{191}$$

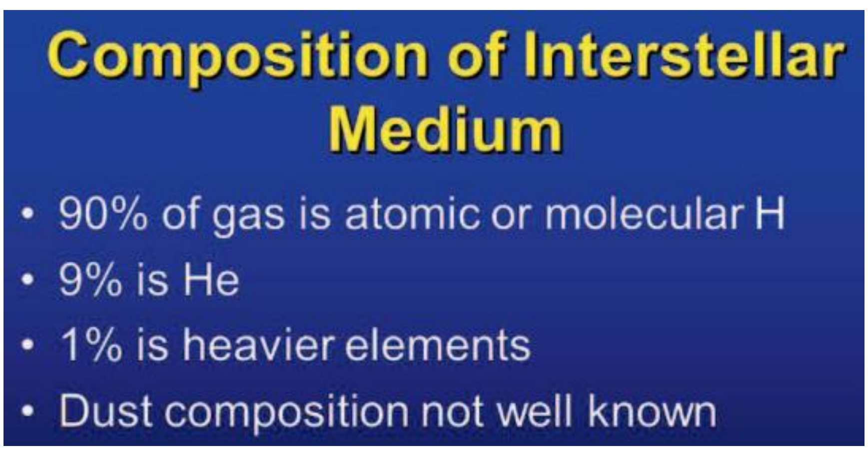

FIG. 4. Composition of the Interstellar Medium IM (Source: Internet)

Appendix H: Composition of the Interstellar Medium IM

Appendix I: A generic procedure to compute extrinsic curvatures and energy densities with lapse functions using mainly the Shoshany Snodgrass work

We will start with the 4D ADM metric where N stands for the lapse function β^i and β^j are shift vectors and δ_{ij} or γ_{ij} are the induced and diagonalized 3D ADM metrics in agreement with the Appendix E. see eq 5.1 pg 15, eq 8.1 pg 33 in [34]. see also eq 21.40 pg [507(b)] [528(a)] in [11], eq 2.123 pg [45(b)] [65(a)] in [36]

$$ds^2 = -N^2 dt^2 + \delta_{ij}(dx^i - \beta^i dt)(dx^j - \beta^j dt) \tag{192}$$

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i - \beta^i dt)(dx^j - \beta^j dt) \tag{193}$$

The 4D ADM matrices are given by: see eqs A.1 and A.2 pg 34 in [34] see also eq 21.42 pg [507(b)] [528(a)] and 21.44 pg [508(b)] [529(a)] in [11],eqs 2.119 and 2.122 pg [45(b)] [65(a)] in [36],eqs 2.2.5 and 2.2.6 pg [67(b)] [82(a)] in [12]

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -N^2 + \beta_k \beta^k & -\beta_j \\ -\beta_i & \gamma_{ij} \end{pmatrix} \tag{194}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{N^2} & -\frac{\beta^j}{N^2} \\ -\frac{\beta^i}{N^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{N^2} \end{pmatrix} \tag{195}$$

The components of the normal vector are given by: see eq A.3 pg 34 in [34]

$$n_\mu = (-N, 0), n^\mu = \frac{1}{N}(1, \beta), \tag{196}$$

In the contravariant matrix above $g^{\mu\nu}\gamma^{ij}$ is the inverse of the 3D induced metric and $\beta_i \equiv \gamma_{ij}\beta^j$.

The 3D induced and diagonalized metrics are $g_{ij}=\gamma_{ij}=\delta_{ij}$ and the shift vectors β^i and β^j refers to the 3D scripts i, j=1, 2, 3. The 4D ADM scripts are μ, ν=0, 1, 2, 3. Now we replace μ→i, ν→j, in the components of the normal vector and use $n_i=0$ since i, j=1, 2, 3 and nμ = (−N, 0) with 0 corresponds to the scripts i, j=1, 2, 3 to get the equation of the extrinsic curvature K_{ij} for the 3D scripts i, j = 1, 2, 3 as a covariant derivative of $n_i=0$. see eq B.12 pg 36 in [34]

$$K_{ij} = \nabla_{(i} n_{j)}. \tag{197}$$

see eq B.13 pg 36 in [34]

$$K_{ij} = \partial_{(i} n_{j)} - \Gamma^\rho_{(ij)} n_\rho \tag{198}$$

Since i, j=1, 2, 3 the scripts of the 3D induced metric and $n_i=n_j=0$ the ordinary symmetrical derivative $\partial(i^n j)$ vanishes and the Christoffel symbol becomes the dominant term. Then the component of the normal vector $n_\mu=(-N, 0)$ that accounts is the term $n_0=(-N)$ with the 4D script μ=0 and the contravariant script of the Christoffel symbol must only be 0.

The equation now becomes:

$$K_{ij} = 0 + N\Gamma^0_{ij} \tag{199}$$

The expansion of the Christoffel symbol gives:

$$K_{ij} = \frac{1}{2} N \left(g^{00}(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}) + g^{0k}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}) \right) \tag{200}$$

Inserting the components of the 4D ADM matrices

$g_{0j}=-\beta_j$, $g_{0i} = -\beta_i$,$g^{00} =- 1/N^2$, $g^{0k} =(-1/N^2\beta^k)$ in the expression above we get the following expression:

$$K_{ij} = \frac{1}{2}N\left(-\frac{1}{N^2}(\partial_i(-\beta_j) + \partial_j(-\beta_i) - \partial_0\delta_{ij}) + (-\frac{1}{N^2}\beta^k)(\partial_i\delta_{kj} + \partial_j\delta_{ki} - \partial_k\delta_{ij})\right) \tag{201}$$

The equation reduces to:

see eqs B.13 pg 36 and 4.28 pg 13 in [34] see also eqs 9 and 10 pg 5 in [1],eq 21.67 pg [513(b)],[534(a)] in [11],eq 2.3.12 pg [71(b)],[86(a)] in [12],eq 2.128 pg [46(b)],[66(a)] in [36]

$$K_{ij} = \frac{1}{2}\frac{1}{N}\left((\partial_i\beta_j + \partial_j\beta_i) + 0\right) = \frac{1}{N}\partial_{(i}\beta_{j)}, \tag{202}$$

$$K_{ij} = \frac{1}{2}\frac{1}{N}\left(\partial_i\beta_j + \partial_j\beta_i\right) = \frac{1}{N}\partial_{(i}\beta_{j)}, \tag{203}$$

Note that for a lapse function N=1 the equation of the extrinsic curvature reduces to:

$$K_{ij} = \frac{1}{2}\left(\partial_i\beta_j + \partial_j\beta_i\right) = \partial_{(i}\beta_{j)}, \tag{204}$$

The trace of the extrinsic curvature is given by $K=K^{\rho}{}_{\rho}=K^{i}{}_{i}$. see pg 14 in [34] and the factor 1 N appears in the trace. Considering the square of the trace the corresponding factor would then be $1/N^2$. Also the term $K_{ij}K^{ij}$ have a factor $1/N^2$ too.

The energy density in function of the extrinsic curvatures is given by: see eq 4.33 pg 14 in [34] [21]

$$\rho \equiv \frac{1}{8\pi G}G_{nn} = \frac{1}{16\pi G}(K^2 - K_{ij}K^{ij}) \tag{205}$$

Compare with the equation of the energy density given in pg 3 in [2].22 see also eq 4.33 pg 14 in [34],eqs 21.162(a) to 21.162(c) pg [552(b)] [573(a)] in [11],eqs 2.88 to 2.90 pg [40(b)] [60(a)] in [36],eq 10.2.30 pg [259(b)] [266(a)] in [23],eqs 2.4.5 to 2.4.6 pg [72(b)] [87(a)] in [12].

$$\rho = T_{ab}n^a n^b = \frac{1}{16\pi}\left({}^{(3)}R + (K^i{}_i)^2 - K_{ij}K^{ij}\right) = \frac{1}{16\pi}\left(\theta^2 - K_{ij}K^{ij}\right) \tag{206}$$

With the trace of the extrinsic curvatures or the expansion of the normal volume elements being $\theta=K=K^{\rho}\rho=K^{i}i$ then $(K^{i}{}_{i})^2=\theta^2=K^2$ and (3) R=0

Generic equations for the energy density where the factor $1/N^2$ and ordinary derivatives of the shift vectors appears are given by: see eqs 5.3 and 5.4 pg 15 in [34].23

$$\rho = \frac{1}{16\pi G}\frac{1}{N^2}((\partial_i\beta_i)^2 - \partial_i\beta_j\partial_i\beta_j), \tag{207}$$

$$\rho = \frac{1}{16\pi G}\frac{1}{N^2}\left(\partial_i(\beta_i\partial_j\beta_j - \beta_j\partial_j\beta_i) - \partial_{[i}\beta_{j]}\partial_{[i}\beta_{j]}\right), \tag{208}$$

Appendix J: Mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original 3+1 ADM Formalism according to MTW and Alcubierre using a lapse function α

This Appendix is a continuation of the Appendix E except for the fact that we do not suppress the lapse function here. Combining the eqs (21.40), (21.42) and (21.44) pgs [507, 508(b)] [534, 535(a)] in [11] with the eqs (2.2.5) and (2.2.6) pgs [67(b)] [82(a)] in [12] using the signature (−, +, +, +) we get the original equations of the 3 + 1 ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{209}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{210}$$

The components of the inverse metric are given by the matrix inverse:

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{211}$$

The space time metric in 3+1 is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{212}$$

But since $dl^2=\gamma_{ij}dx^idx^j$ is the ADM induced metric and must be a diagonalized metric then $dl^2=\gamma_{ii}dx^idx^i$ and we have

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{213}$$

Expanding the square term and recombining all the terms we have:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{214}$$

Note that the expression above is exactly the eq (2.2.4) pgs [67(b)] [82(a)] in [12]. It also appears as eq 1 pg 3 in [1]. Changing the signature from (−, +, +, +) to signature (+, −, −, −) we have:

$$ds^2 = -(-\alpha^2 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{215}$$

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{216}$$

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{217}$$

We can see that $\beta^i=-X^i$, $\beta_i=-Xi$ and $\beta_i\beta^i=X_iX^i$ with Xi as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector. Hence we have:

$$ds^2 = (\alpha^2 - X_iX^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{218}$$

Then the equation of the Natario warp drive spacetime with a constant speed vs in the original 3+1 ADM formalism with a lapse function is given by:

$$ds^2 = (\alpha^2 - X_iX^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{219}$$

Inserting the components of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^\theta rs d\theta \tag{220}$$

We have:

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drsdt + X_\theta d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{221}$$

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{222}$$

The lapse function is equal to 1 inside and outside the Natario warp bubble while having large values in the Natario warped region.

Remarks

References [11],[12],[13],[14],[15],[22],[23],[24],[25],[26],[27],[28] and [36] are standard textbooks used to study General Relativity or warp drive spacetimes and these books are available or in paper editions or in electronic editions all in Adobe PDF Acrobat Reader.

We have the electronic editions of all these books

In order to make easy the reference cross-check of pages or equations specially for the readers of the paper version of the books we adopt the following convention: when we refer for example the pages [507, 508(b)] or the pages [534, 535(a)] in [11] the (b) stands for the number of the pages in the paper edition while the (a) stands for the number of the same pages in the electronic edition displayed in the bottom line of the Adobe PDF Acrobat Reader.

All the numerical plots presented in this work are available for those interested to check out our results. We gladly provide the files. These plots also present the results mentioned in [18].

We used the first version of Barak Shoshany and Ben Snodgrass work arXiv:2309.10072v1 (gr-qc) 18 Sep 2023 to assemble our own work. We will adapt our work if future versions of their work appears.

Epilogue

- The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C. Clarke [24].
- The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein [25, 26].